Some of Douglas Munn's Contributions to Representation Theory of Semigroups

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Outline

Introduction I

- **Early works:**
  - In 1933, Suschkewitch.
  - Since then, Clifford, Munn, Ponisovsky, Lallement, Petrich, Preston and McAlister.
- **Douglas Munn's Work:**
  - In 1955, he completed his PhD thesis.
  - In the period (1955-1962), he wrote 6 papers about this theory.

Introduction II

Munn's Papers:

[1] *On Semigroup Algebras*: Proc. Camb. Phil. Soc. 51, 1-15(1955).

[2] *Matrix Representations of Semigroups*: Proc. Camb. Philos. Soc. 53, 5-12(1957).

[3] *Characters of The Symmetric Inverse Semigroup*: Proc. Camb. Philos. Soc. 53, 13-18(1957).

[4] *Irreducible Matrix Representations of Semigroups*: Q. J. Math. 11, 295-309(1960).

[5] *A Class of Irreducible Matrix Representations of An Arbitrary Inverse Semigroup*: Proc. Glasg. Math. Assoc. 5, 41-48(1961).

[6] *Matrix Representations of Inverse Semigroups*: Proc. Lond. Math. Soc. 14, 165-181(1964).

Main theme

Connect the representations of a semigroup to the representations of certain associated groups.

Introduction III

Basic Definitions:

Representation of a semigroup

Let V be a vector space of dimension n over a field F. A representation Γ of a semigroup S of degree n over F is a homomorphism from S to $End(V)$, the semigroup of all linear transformations of V over F.

Irreducible representation

Let V be a representation space for Γ. Then V and Γ are irreducible S-representation if and only if the only invariant subspaces of V are $\{0\}$ and V itself (reducible otherwise).

Equivalent representations

Two S-representations Γ and Λ are equivalent $\iff \exists$ an isomorphism α such that $\forall\ a \in S$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & V \\ {\scriptstyle \Gamma(a)}\downarrow & & \downarrow{\scriptstyle \Lambda(a)} \\ V & \xrightarrow{\alpha} & V \end{array}$$

On Semigroup Algebras (1955) I

When a semigroup S is semisimple?

Semisimplicity

A principal series of a semigroup S is a chain

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \emptyset$$

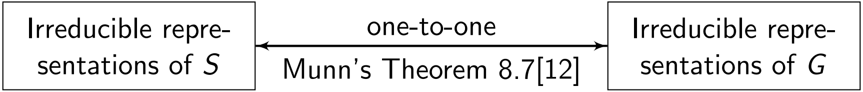
of ideals S_i of S $(i = 1, \ldots n)$, and such that S_i is maximal in S_{i-1}. The Rees quotients S_i/S_{i+1} are called the principal factors of S and they are either (0)-simple or null. Then the semigroup S is called semisimple if it has a principal series and every principal factor of S is simple.

On Semigroup Algebras (1955) II

Munn's Theorem 8.7[12]

- Let $S = S_{mn}[G, P]$ and $\{\Gamma_i; i = 1, \ldots, k\}$ be a complete set of inequivalent irreducible representations of G over F whose characteristic is zero or a prime not dividing the order of G.
- Let the algebra of S be semisimple.
- Then $\{\Gamma_i'; i = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of S over F, where Γ_i' is the basic extension of Γ_i.

Clifford's construction of the representations of Rees matrix semigroup $S_{mn}[G, P]$:



Irreducible representations of inverse semigroup:

(1) Let S be an inverse semigroup. Assume that S has a principal series.

(2) Let $\{e_{ij}; j = 1, \ldots, m_i\}$ be the set of non-zero idempotents of S_i/S_{i+1} $(i = 1, \ldots, n)$.

(3) Let F be a field of characteristic zero or a prime not dividing the order of any of the basic groups of any of the principal factors S_i/S_{i+1}.

(4) Let $\{\gamma'_{ir}; r = 1, \ldots, k_i\}$ be a complete set of inequivalent irreducible representations of S_i/S_{i+1} over F.

(5) Define the mapping γ^*_{ir} on S by the rule

$$\gamma^*_{ir}(x) = \sum_{j=1}^{m_i} \gamma'_{ir}(x^\theta e_{ij}),$$

where θ is the natural homomorphism of S onto S/S_{i+1}.

(6) Then $\{\gamma^*_{ir}; i = 1, \dots, n; r = 1, \dots, k_i\}$ is a complete set of inequivalent irreducible representations of S over F.

The Characters of The Symmetric Inverse Semigroup(1957)

The characters of irreducible representations of the symmetric inverse semigroup I_n are expressible as sums of the characters of irreducible representations of the symmetric groups S_r $(r = 0, \ldots, n)$ over a field F with characteristic zero.

Irreducible Matrix Representations of Semigroups I (1960)

Definitions:

- A semigroup S is said to have a minimal condition M_f on the principal ideals if every set of principal ideals of S has a minimal member.
- Let Γ be a representation of S. $V(\Gamma) = \{x \in S : \Gamma(x) = 0\}$.
- Γ is called *principal* if $S - V(\Gamma)$ contains a unique minimal $\mathcal{J}$-class of S. This $\mathcal{J}$-class J is called the *apex* of Γ.
- The principal representation Γ is described by the rule:

$$\Gamma(x) \neq 0 \Longleftrightarrow J \leqslant J_x \ ,$$

where J_x is the $\mathcal{J}$-class of x.

Irreducible Matrix Representations of Semigroups II (1960)

The main results:

- There is a 1-1 correspondence between the irreducible principal representations of S and the irreducible representations vanishing at zero of the (0-)simple principal factors of S.
- **For a semigroup S satisfying the minimal condition M_f on its principal ideals,** then every irreducible representation of S is principal.

A Class of Irreducible Matrix Representations of An Arbitrary Inverse Semigroup (1961) I

The maximal group homomorphic image of an inverse semigroup:
Let S be an inverse semigroup and let a relation σ be defined on S by the rule that

$$x\sigma y \iff \exists \text{ an idempotent } e \in S \text{ such that } ex = ey.$$

Then we have:

1. σ is a congruence relation and S/σ is a group.
2. If τ is any congruence on S with the property that S/τ is a group, then $\sigma \subseteq \tau$ and so S/τ is isomorphic with some quotient group of S/σ. The quotient S/σ is called the maximal group homomorphic image of S and is denoted by G_S.

A Class of Irreducible Matrix Representations of An Arbitrary Inverse Semigroup (1961) II

Prime representations of an inverse semigroup:

- If the vanishing set $V(\Gamma)$ is empty or a prime ideal, then the representation Γ is called a *prime* representation of S.
- Let S be an inverse semigroup and F be a field:
  1. Let Γ be a prime irreducible representation of S over F and let V=$V(\Gamma)$. Then $S \setminus V$ is an inverse semigroup and

$$\Gamma(x) = \begin{cases} \Gamma^*(\bar{x}) & \text{if } x \in S \setminus V, \\ 0 & \text{if } x \in V, \end{cases}$$

     where $x \to \bar{x}$ is the natural homomorphism of $S \setminus V$ onto $G_{S \setminus V}$, and Γ^* is an irreducible representation of $G_{S \setminus V}$.
  2. Let V be the empty set or a prime ideal of S. Then $S \setminus V$ is an inverse semigroup. Also, if Γ^* is any irreducible representation of $G_{S \setminus V}$, then the mapping Γ is a prime irreducible representation of S.

Thank You!

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